Bounds for blocking time in a queuing system with an unreliable server

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Abstract

The characterization of the distributions by their qualitative properties made it possible to study characteristics of the systems such as: the mean stationary waiting time in the GI/GI/1 system and the mean time of life system constituted by two repairable elements *"S. Adjabi, K. Lagha and al. (2003)"*.
In this work we consider a single-server queuing system with an unreliable server and service repetition. We consider exponential server failure time and service time belonging to a non parametric class (IFR, NBU, DFR and NWU class). Then we obtain bounds for the blocking time in the system by using lower and upper bounds of reliability functions presented by *"Sengupta (1994)"*.
Key words: Bounds, Nonparametric laws, blocking time, failure .

1 Introduction

Consider a single-server queuing system with an unreliable server and service repetition. If a customer is being served at the instant of server failure, the service is interrupted, is restarted anew immediately (the repair time is assumed to be instantaneous) and ends whenever the service period is failure free.
The total time taken by a customer from the instant he enters for service to the instant when he ends his service is called the "blocking time", which can be represented by:

$$Z_\lambda = X.\mathbf{1}_{\{X \le Y\}} + (Y + Z^*_\lambda).\mathbf{1}_{\{X > Y\}}, \quad \lambda > 0. \tag{1}$$

Where X, Y and Z^*_λ are independent non-negative random variables. X is the service time with free interruption, $\mathcal{L}_X(\lambda) = E(e^{-\lambda X})$ be the Laplace-Stieltjes Transform of the random variable X. Y is the server failure time and is assumed to have an exponential distribution with mean $1/\lambda$. Z^*_λ has the same distribution as Z_λ (denoted by $Z^*_\lambda \overset{d}{=} Z_\lambda$), and $\mathbf{1}_{\{X \le Y\}}$ is the indicator function of the event $\{X \le Y\}$ defined as 1 if the event $\{X \le Y\}$ occurs and 0 otherwise. If Y is not exponential, we substitute Z instead of Z_λ in the equation (1).
The above queuing system is introduced by *"B. Dimitrov, Z. Khalil (1990)"*, based on the blocking time Z_λ. Dimitrov and Khalil (1990) obtained two characterizations of the service time exponential distribution. *"XU. JIAN-LUN (1998)"* obtained another characterization via the coefficient of the variation of Z_λ.

Instead of characterizing the exponentiality of the service time, we consider a general service time but having a qualitative property. This approach made it possible to study characteristics of many systems *"S. Adjabi, K. Lagha and al. (2003)"*. In this paper we determine the upper or lower bounds for the mean blocking time in the system, in the theorems (1), (2), (3) and (4) of the section (2).

2 Main results

Let $F(t)$, $G(t)$ and $R(t)$ denote the cumulative distribution functions (cdf) of Z, X and Y, respectively. Let $\overline{F}(t) = 1 - F(t)$, $\overline{G}(t) = 1 - G(t)$ and $\overline{R}(t)$, $0 \le t < \infty$.

Lemme 1 *Suppose that X is not degenerate at point zero and Z defined as (1), so*

$$EZ = E(\min(X,Y))/p. \tag{2}$$

Where, $p = \int_0^\infty G(t)dR(t) = P(X \leq Y)$.

Indeed:
X, Y and Z are independent. It can be seen from (1) with a simple probabilistic argument that,

$$\overline{F}(t) = \overline{G}(t)\overline{R}(t) + \int_0^t \overline{F}(t-u)\overline{G}(u)\, dR(u), \ \ \forall t \geq 0. \tag{3}$$

We introduce $M = \min(X,Y)$ v.a. Then,

$$EM = \int_0^\infty P(\min(X,Y) \geq t)\, dt = \int_0^\infty \overline{G}(t)\overline{R}(t)\, dt.$$

From (3),

$$EZ = \int_0^\infty \overline{F}(t) = EM + EZ\int_0^\infty \overline{G}(t)dR(t).$$

From where,

$$\int_0^\infty G(t)dR(t).EZ = EM.$$

Let $p = \int_0^\infty G(t)dR(t)$ then,

$$p = \int_0^\infty P(X \leq Y \leq t)\, dt = P(X \leq Y).$$

If X is not degenerate at point zero we have $EZ = EM/p$.

Remark 1
If Y have an exponential distribution with mean $1/\lambda$, then

- $E(\min(X,Y)) = \frac{1}{\lambda}(1 - \mathcal{L}_X(\lambda))$.
- $P = P(X \leq Y) = \mathcal{L}_X(\lambda)$.

So,

$$EZ_\lambda = \frac{1 - \mathcal{L}_X(\lambda)}{\lambda\mathcal{L}_X(\lambda)}.$$

Assume that distribution service time belongs to a common family (e.g. IFR, DFR, NBU, NWU) with a know moment. The upper and lower bounds on the mean blocking time are derived.

Theorem 1 *Suppose that X is not degenerate at point zero of the rth moment EX^r and Y be exponential with mean $1/\lambda$. X belongs to DFR implies*

$$EZ_\lambda \leq \frac{x_0(e^r - e^{-\lambda x_0}) + (r + \lambda x_0)x_0^r I_r}{re^r + \lambda x_0 e^{-\lambda x_0} - \lambda(r + \lambda x_0)x_0^{-r} I_r},$$

where $I_r = \int_{x_0}^{+\infty} t^{-r}e^{-\lambda t}dt$ *and* $x_0 = r\left[\frac{EX^r}{\Gamma(r+1)}\right]^{1/r}, r > 0$.

proof
The cdf G of X belongs to DFR *"Sengupta (1994)"* $\Rightarrow$

$$\overline{G}(x) \leq \begin{cases} e^{-(rx/x_0)} & ,x < x_0 \\ \\ (x_0/x)^r e^{-r} & ,x \geq x_0 \end{cases} \quad , \quad \text{with} \quad x_0 = r\left[\frac{EX^r}{\Gamma(r+1)}\right]^{1/r}.$$

Then,

$$E(\min(X,Y)) = \int_0^\infty \overline{G}(t)e^{-\lambda t}dt \leq \frac{x_0}{r+\lambda x_0}\left(1 - e^{-(r+\lambda x_0)}\right) + e^{-r}x_0^r\int_{x_0}^{+\infty} t^{-r}e^{-\lambda t}dt,$$

and using lemma (1) we derive

$$EZ_\lambda \leq \frac{x_0(e^r - e^{-\lambda x_0}) + (r+\lambda x_0)x_0^r I_r}{re^r + \lambda x_0 e^{-\lambda x_0} - \lambda(r+\lambda x_0)x_0^{-r}I_r},$$

where $I_r = \int_{x_0}^{+\infty} t^{-r}e^{-\lambda t}dt$ converge for all $x_0 > 0$ and $r > 0$.
Particular case If $r = 1$ and $x_0 = 1/\lambda$ then,

$$EZ_\lambda \leq \frac{1}{\lambda}\left[\frac{\sinh 1 + I_1}{\cosh 1 - I_1}\right], \quad \text{and} \quad I_1 = \int_{1/\lambda}^{+\infty} t^{-1}e^{-\lambda t}dt.$$

Theorem 2 *With the same conditions than theorem 1, if X belongs to IFR than*

$$EZ_\lambda \geq x_0\left[\frac{1 - e^{-(1+\lambda x_0)}}{1 + x_0\lambda e^{-(1+\lambda x_0)}}\right], \quad x_0 = EX.$$

proof
The cdf G of X belongs to IFR *"Sengupta (1994)"* $\Rightarrow$

$$\overline{G}(x) \geq \begin{cases} e^{-t/x_0} & ,t < x_0 \\ \\ 0 & ,t \geq x_0. \end{cases} \quad , \quad \text{tel que} \quad x_0 = EX.$$

Thus,

$$EM = \int_0^\infty \overline{G}(t)e^{-\lambda t}\,dt \geq x_0\left(\frac{1 - e^{-(\lambda x_0 + 1)}}{1 + \lambda x_0}\right).$$

Using lemma (1) we yield results.

Theorem 3 *With the same conditions than theorem 1, if X belongs to NBU than,*

$$EZ_\lambda \geq \frac{\beta + e^{-\beta} - 1}{\lambda(1 - e^{-\beta})}, \quad \beta = \lambda x_0.$$

proof
The cdf G of X belongs to NBU *"Sengupta (1994)"* $\Rightarrow$

$$\overline{G}(x) \geq \begin{cases} 1 - x/x_0 & ,x < x_0 \\ \\ 0 & ,x \geq x_0 \end{cases} \quad , \quad \text{with} \quad x_0 = EX.$$

Thus,

$$EM \geq \frac{1}{x_0\lambda^2}\left(\lambda x_0 + e^{-\lambda x_0} - 1\right).$$

Using lemma (1) we obtain

$$EZ_\lambda \geq \frac{\beta + e^{-\beta} - 1}{\lambda(1 - e^{-\beta})}, \quad \beta = \lambda x_0.$$

Particular case: If $x_0 = 1/\lambda$ and $\beta = 1$ then,

$$EZ_\lambda \geq \frac{e^{-1}}{\lambda(1 - e^{-1})}.$$

Theorem 4 *With the same conditions than theorem 1, if X belongs to NWU than*

$$EZ_\lambda \leq \frac{x_0 e^{\beta} I_1}{1 - \beta e^{\beta} I_1}, \quad \beta = \lambda x_0 \text{ and } I_1 = \int_{x_0}^{+\infty} t^{-1} e^{-\lambda t} dt.$$

proof
The cdf G of X belongs to NWU *"Sengupta (1994)"* $\Rightarrow$

$$\overline{G}(t) \leq \frac{x_0}{t + x_0}, \quad t \geq 0, \quad \text{and} \quad x_0 = EX.$$

Then,

$$EM \leq x_0 e^{\lambda x_0} I_1.$$

Where I_1converge and using lemma (1), we yield result for $\beta = \lambda x_0$.

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